Computability

CS70: Discrete Mathematics and Probability Theory

UC Berkeley – Summer 2026

Lecture 15

Ref: Note 12

Uncomputable Functions

From last time - are these countable or uncountable?

(A) Set of all predicates?

(B) Set of all programs?

So: No bijection possible between programs and predicates
$\Longrightarrow$ Some predicates have no program to compute them
(*In fact: Uncountably many predicates have no program!*)

Last Lecture: Uncomputable predicates (functions) exist

This Lecture: Do *interesting/practical* functions exist with no program that can compute them?

We'll see the answer is Yes!

A Deep Problem: Self Reference

From Lecture 1!

Is this a proposition? “This statement is false.”

If it’s false then it’s true...
if it’s true it’s false...
Liar’s paradox

Russell’s Paradox:
Does the set of all sets which do not contain themselves contain itself?

$R =$ set of all sets that are not members of themselves
Math-y: $R = \{x \mid x \notin x\}$
$R \in R$? Then $R \notin R$
$R \notin R$? Then $R \in R$

Important question: Were mind-altering substances involved?

Issue confounding both Liar’s Paradox and Russell’s Paradox:
Self-reference...

An Interesting/Practical Problem

Problem: Write a program that analyzes an input program P to determine if it halts on some program input x (or if it goes into an infinite loop).

Wait... a program that takes a program as input? Is that a thing?
Of course: That's what a compiler is!
That's what a static analysis tool is!

Specifically, I want the following function (which *always* gives an answer):

$HALT(P, x)$: P is a program, x is an input to P
Returns "Yes" if P halts when run on x
"No" if P loops forever when run on x

How do we provide a program as input to a program?
Seems weird at first, but it's really not.... a program is just a string of bits
Bits are bits – they have no "meaning" until we give them meaning
One byte: `01000001`
Is it a number? Could be... could represent 65
Is it a character? Could be... could be ASCII character 'A'
Is it code? could be... Intel 8080 instruction "`MOV B,C`"

Implementing HALT

$HALT(P,x)$: P is a program, x is an input to P

Returns "Yes" if P halts when run on x
"No" if P loops forever when run on x

Solution: Just run the program! – in pseudo-Python....

Use Python `eval` function to evaluate $P(x)$

```
if P(x) halts:
  return "Yes"
else:
  return "No"
```

Correct? Incorrect? Maybe?

Incorrect! If $P(x)$ loops forever, never get to the `if`...

Run for a million steps? Maybe it would halt in 2 million....
Run for a day? Maybe it would halt in 2 days....

Bottom line: Can't ever stop and say "that was enough"

No Program Exists for HALT – Part 1

Theorem: There is no program that correctly implements $HALT$.

Proof: Assume for the sake of contradiction that we have a program $HALT(P,x)$ that always gives the right answer ($HALT$ always halts!).

Use $HALT$ to define a new function/program:

Turing(P):
1. If HALT(P,P) ="Yes": Go into an infinite loop
2. Else: halt immediately

Possible? Yes: $HALT$ exists by assumption – string of bits defines $HALT$
Turing exists? It's above – just more bits

Recap to now:
Assume $HALT$ function/program exists – it's a string of bits (a file?)
Can add a new function to program, Turing – just more bits

Have bits – data – that is full implementation of Turing

Can do anything with those bits that we can do with any other data
... including passing those bits as data to a program!

No Program Exists for HALT – Part 2

Turing(P):

1. If HALT(P,P) ="Yes": Go into an infinite loop
2. Else: halt immediately

Question: Does Turing(Turing) halt?

Yes?

Turing(Turing) calls *HALT*(Turing, Turing) which returns "Yes"

$\Longrightarrow$ Turing(Turing) goes into an infinite loop and doesn't halt.

No?

Turing(Turing) calls *HALT*(Turing, Turing) which returns "No"

$\Longrightarrow$ Turing(Turing) halts

So if *HALT* works, then ...

Turing(Turing) halts $\Longrightarrow$ Turing(Turing) doesn't halt

Turing(Turing) doesn't halt $\Longrightarrow$ Turing(Turing) halts

Impossible! Contradiction! No program can correctly implement *HALT*. $\square$

Confused? That's normal. Questions?

Another View: Diagonalization

Fact 1: Any program is a fixed length string.
Fact 2: Fixed length strings are enumerable.
Fact 3: Program inputs are finite length strings (so enumerable).
Fact 4: A program halts or doesn't for any input (even if we don't know which)

Enumeration of programs (in rows below) – inputs as columns – Halt or Loop:

	P_1	P_2	P_3	$\cdots$
P_1	H	H	L	$\cdots$
P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Important: We're not saying we can construct this table. Only that it exists!

Create "diagonal program" – flips H and L
Turing is the diagonal program

Turing is a program, so must be in the enumeration of all programs
... but it halts/loops differently than any program for at least 1 input
... so it's different from any program on the list
Contradiction!

So Turing can't be a program....

Take a Breather...

Question: What are programs?

(A) Instructions

(B) Text

(C) Binary String

(D) They run on computers

A Technical Break For Some History

Question: "Turing"??? What's that? Who is that?

This proof was invented by Alan Turing in 1937
- "On Computable Numbers, with an Application to the Entscheidungsproblem"
- The *what*?

David Hilbert (1862–1943): Mathematician and Philosopher
- Famous for posing challenge problems – 23 famous problems in 1900
  - *Similar: Seven "millennium problems" from the Clay Institute in 2000*
- Hilbert posed the Entscheidungsproblem in 1928
- Asks if a process/algorithm exists to say whether a statement is valid
  - 1936: Turing (and Alonzo Church): No

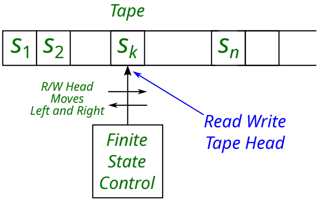
What's an algorithm? What's a computer? Can computation be mechanized?
- All non-obvious in 1928

Alan Turing: Father of Computer Science
- Defined abstract computing machine – now called a Turing Machine
- Designed machines that broke codes in World War II
- Talked about artificial intelligence and a test now called the "Turing Test"
- Namesake of the top prize in Computer Science: The Turing Award
- Tragic life and victim of the time in which he lived... Movie: *The Imitation Game*

Turing Machines

How does someone envision a programmable "computing machine"?

A "Turing Machine" (TM):



The machine definition (Finite State Control) is an algorithm

A Universal Turing Machine's algorithm simulates another TM
    Input (on tape) is the encoding of a TM (a program)
    Halting problem: Ask if input TM halts on given input

If a machine/language can simulate a Turing machine, it is "Turing Complete."

Church-Turing Thesis: Any problem that can be solved with a computational procedure can be solved by a Turing machine.

Reductions from HALT

A reduction from problem A to problem B uses a solution to B to solve A
Example: Reduce “compute multiplicative inverse” to “compute Ext-GCD”

Problem: Write a program analyzer to determine if $P(x)$ outputs “Hello world”
Call this problem HW-CHECK(P, x)

Idea: Let’s reduce HALT to HW-CHECK

Program:
$HALT(P, x)$
Transform P to P':
Remove existing “print” statements
Add print “Hello world” at end (at program termination)
Call HW-CHECK(P', x)
Return result

By construction, $P'(x)$ prints “Hello world” if and only if P halts on x
So if HW-CHECK works correctly, then this solves HALT.

But no program can solve HALT! Contradiction!
$\Longrightarrow$ No program can solve HW-CHECK correctly. $\square$

Reductions

A reduction from problem A to problem B uses a solution to B to solve A
Example: Reduce "compute multiplicative inverse" to "compute Ext-GCD"

Notation: We write $A \leq B$ to say there is a reduction from A to B
Usual way of saying: "A is reducible to B"

Complication: There are different kinds of reductions... not important here

Theorem: If $A \leq B$ and A is uncomputable, then B is uncomputable.

Proof idea: Assume (for contradiction) B is computable, so a program for B exists. Use reduction with B's program to solve A. But that's impossible... $\square$

On the previous slide:
Showed $HALT \leq$ HW-CHECK (a program for $HALT$ using a program for HW-CHECK)
We know HALT is uncomputable
$\implies$ HW-CHECK is uncomputable

Undecidable Problems 1

In Theory of Computation, predicates are often called "decision problems"
An uncomputable decision problem is "undecidable"

Undecidable problems about programs:
Given program P and input x, does P halt when run on x?
Given program P and input x, does P print "hello world" when run on x?
Given program P and input x, does P execute "line n" when run on x?
Given program P and input x, does P access an array out of bounds?

All proofs are similar to $HALT \leq$ HW-CHECK reduction given earlier

Rice's Theorem: Deciding if *any* non-trivial property is met during the execution of program P on input x is undecidable.

"Static Analysis" tools check programs for bugs, security vulnerabilities, etc.
Rice's Theorem says making a perfect static analysis tool is impossible!

Undecidable Problems 2

Problems that aren't (or don't seem like!) analyzing programs.

Post Correspondence Problem

Given "domino" types with labels, unlimited copies of each

Can copies be lined up to read same top/bottom?

$$\left\{\left[\frac{\mathtt{b}}{\mathtt{ca}}\right],\left[\frac{\mathtt{ca}}{\mathtt{a}}\right],\left[\frac{\mathtt{a}}{\mathtt{ab}}\right],\left[\frac{\mathtt{abc}}{\mathtt{c}}\right]\right\} \Longrightarrow \left[\frac{\mathtt{a}}{\mathtt{ab}}\right]\left[\frac{\mathtt{b}}{\mathtt{ca}}\right]\left[\frac{\mathtt{ca}}{\mathtt{a}}\right]\left[\frac{\mathtt{a}}{\mathtt{ab}}\right]\left[\frac{\mathtt{abc}}{\mathtt{c}}\right]$$

Solving Diophantine Equations

Algorithms exist to approximate roots of polynomial equations over $\mathbb{R}$

But what about over $\mathbb{Z}$?

Example: Are there integer roots to $x^n + y^n = 5$?

Conway's "Game of Life" (cellular automaton – pattern evolution)

Grid of locations with evolving state – based of neighbors

Stephen Wolfram's book "A New Kind of Science"

Billiard Ball Simulation

Given balls in an area with perfectly elastic collisions

Does a ball go in a pocket?

Back To Logic

Logic: Start with axioms, derive other propositions

Gödel:

Any set of axioms is either

inconsistent (can prove false statements) or

incomplete (there are true statements that cannot be proven).

Proof idea:

Formal system F: axioms, rules of inference, ...

Consider $S(F) =$ "This statement is not provable in F"

Provable (that's it's not provable)? Inconsistent

Not provable? Incomplete

Can prove using undecidability of HALT too! (see notes)

Concrete example:

Continuum hypothesis: "no cardinality between naturals and reals"

Continuum hypothesis not disprovable in ZFC (Gödel 1940)

Continuum hypothesis not provable (Cohen 1963)

Lecture 15 Summary

The problem with self-reference
- Liar's Paradox, Russell's Paradox, ...

Uncomputable functions
- Programs as data
- The halting problem
  - Proof of undecidability
  - Diagonalization view of the proof

Some History
- Alan Turing
- David Hilbert

Reductions: Solving one problem with another
- Undecidability of basic program behavior
- HALT reduction to "prints hello world"

Some additional uncomputable functions

Gödel's Incompleteness Theorem

Starting Next Week: Probability

Next up? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?

Count blue. Count total. Divide.

It might get a little more complicated than that...

See you next time!