CS 70 Discrete Mathematics and Probability Theory
Summer 2026 Tate

HW 04

Due: Saturday, 07/25, 4:00 PM
Grace period until Saturday, 07/25, 6:00 PM
Remember to show your work for all problems!

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Past Probabilified

Note 13

In this question we review some of the past CS70 topics, and look at them probabilistically.
For the following experiments, define an appropriate sample space Ω, and give the probability function $\mathbb{P}[\omega]$ for each $\omega \in \Omega$. Then compute the probabilities of the events E_1 and E_2.

(a) Fix a prime $p > 2$, and uniformly sample twice with replacement from $\{0,\ldots,p-1\}$ (assume we have two $\{0,\ldots,p-1\}$-sided fair dice and we roll them). Then multiply these two numbers with each other in $(\bmod p)$ space.

$$E_1 = \text{The resulting product is } 0.$$
$$E_2 = \text{The product is } (p-1)/2.$$

(b) Make a graph on n vertices by sampling uniformly at random from all possible edges, (assume for each edge we flip a coin and if it is head we include the edge in the graph and otherwise we exclude that edge from the graph).

$$E_1 = \text{The graph is complete.}$$
$$E_2 = \text{vertex } v_1 \text{ has degree } d.$$

2 Is This EECS 126?

Note 13
Note 14

Edward loves birdwatching. There are three sites he birdwatches at: Site A, B, and C. He goes to Site A 60% of the time, Site B 35% of the time, and Site C 5% of the time. The probability of seeing a nightingale at each site is $\frac{1}{10}$, $\frac{3}{10}$, and $\frac{2}{5}$, respectively. Using the information above, answer the following questions.

(a) What is the total probability that Edward sees a nightingale?

(b) Given that Edward sees a nightingale, what is the probability that Edward went to site B?

(c) Given that Edward didn't go to site B, what is the probability that Edward doesn't see a nightingale?

3 Presidential Election

Note 13
Note 14

We traveled back in time to 1960 and want to determine which presidential candidate will win this coming election. There are two candidates, John F. Kennedy and Richard Nixon, competing for 50 states.

(a) For each state, Kennedy has probability p of winning. What is the probability that Kennedy will win exactly half the states?

(b) What is the probability that Kennedy will win at least one state?

(c) Say that $p = 0.25$ or $p = 0.75$ with equal chance, so $\mathbb{P}[p = 0.25] = \mathbb{P}[p = 0.75] = 0.5$. If Kennedy wins every single state, what is the probability (we believe) that $p = 0.75$? Note that in reality, either $p = 0.25$ or $p = 0.75$ deterministically, but we are trying to find **how likely was it that** $p = 0.75$ according to our observation.

4 Solve the Rainbow

Note 14

Your roommate was having Skittles for lunch and they offer you some. There are five different colors in a bag of Skittles: red, orange, yellow, green, and purple, and there are 20 of each color. You know your roommate is a huge fan of the green Skittles. With probability $1/2$ they ate all of the green ones, with probability $1/4$ they ate half of them, and with probability $1/4$ they only ate 5 green ones. [There are some calculations in this problem, which you will need the assistance of a calculator. We recommend rounding the solution to at least 3 significant digits, if written as a decimal.]

(a) If you take a Skittle from the bag, what is the probability that it is green?

(b) If you take two Skittles from the bag, what is the probability that at least one is green?

(c) If you take three Skittles from the bag, what is the probability that they are all green?

5 Cliques in Random Graphs

Note 13
Note 14

Consider the graph $G = (V,E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v, we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads.

(a) What is the size of the sample space?

(b) A k-clique in a graph is a set S of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example, a 3-clique is a triangle. Let E_S be the event that a set S forms a clique. What is the probability of E_S for a particular set S of k vertices?

(c) Suppose that $V_1 = \{v_1, \dots, v_\ell\}$ and $V_2 = \{w_1, \dots, w_k\}$ are two arbitrary sets of vertices. What conditions must V_1 and V_2 satisfy in order for E_{V_1} and E_{V_2} to be independent (the events that these vertices form a k-clique)? Prove your answer.

(d) Prove that $\binom{n}{k} \le n^k$. (You might find this useful in part (e)).

(e) Prove that the probability that the graph contains a k-clique, for $k \ge 4\log_2 n + 1$, is at most $1/n$. *Hint:* Use the union bound.

6 Independent Complements

Note 14

Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two independent events.

(a) Prove or disprove: $\overline{A}$ and $\overline{B}$ must be independent.

(b) Prove or disprove: A and $\overline{B}$ must be independent.

(c) Prove or disprove: A and $\overline{A}$ must be independent.

(d) Prove or disprove: It is possible that $A = B$.

7 Pairs of Beads

Note 14

Sally has a set of $2n$ beads ($n \ge 2$) of n different colors, such that there are two beads of each color. She wants to give out pairs of beads as gifts to all the other $n-1$ TAs, and plans on keeping the final pair for herself (since she is, after all, also a TA). To do so, she first chooses two beads at random to give to the first TA she sees. Then she chooses two beads at random from those remaining to give to the second TA she sees. She continues giving each TA he sees two beads chosen at random from her remaining beads until she has seen all $n-1$ TAs, leaving her with just the two beads she plans to keep for herself. Prove that the probability that at least one of the other TAs (*not* including Sally herself) gets two beads of the same color is at most $\frac{1}{2}$.

8 Trick or Treat

Note 15

Ryan and Suraj are trick or treating together, and are each trying to collect all n flavors of Laffy Taffy. At each house, they each receive a Laffy Taffy, chosen uniformly at random from all flavors. However, Suraj will throw a tantrum if Ryan gets a flavor he doesn't, so they agree that if they receive different flavors, they'll both politely return the candy they got, and move on to the next house. What is the minimum number of houses that they have to visit such that the probability of not collecting all flavors is at most $\frac{1}{n^2}$? [The inequality $(1+x) \le e^x$ might be useful with x substituted with the appropriate value.]