CS 70 Discrete Mathematics and Probability Theory
Summer 2026 Tate

HW 01

Due: Saturday, 7/4, 4:00 PM
Grace period until Saturday, 7/4, 6:00 PM
Remember to show your work for all problems!

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Universal Preference

Note 4

Suppose that preferences in a stable matching instance are universal, i.e., all n jobs share the preferences $C_1 > C_2 > \cdots > C_n$ and all candidates share the preferences $J_1 > J_2 > \cdots > J_n$.

(a) What matching do we get from running the propose-and-reject algorithm with jobs proposing? Prove that this happens for all n.

[*Hint*: Start with small examples and go through the algorithm. Do you see a pattern?]

(b) What matching do we get from running the propose-and-reject algorithm with candidates proposing? Explain.

(c) What does this tell us about the number of stable matchings? Justify your answer.

2 Pairing Up

Note 4

Prove that for every even $n \geq 2$, there exists an instance of the stable matching problem with n jobs and n candidates such that the instance has at least $2^{n/2}$ distinct stable matchings.

[*Hint*: It can help to start with some small examples; find an instance for $n = 2$, and think about how you can use these preference lists to construct an instance for $n = 4$. After this, you should be in a good position to generalize the construction for all even n. Additionally, $2^{n/2}$ is a very specific number; try to think about how your construction would build such a number as n increases.]

3 Short Tree Proofs

Note 5

Let $G = (V,E)$ be an undirected graph with $|V| \geq 1$.

(a) Prove that every connected component in an acyclic graph is a tree.

(b) Suppose G has k connected components. Prove that if G is acyclic, then $|E| = |V| - k$.

(c) Prove that a graph with $|V|$ edges contains a cycle.

4 Planarity and Graph Complements

Note 5

Let $G = (V,E)$ be an undirected graph. We define the complement of G as $\overline{G} = (V,\overline{E})$ where $\overline{E} = \{(i,j) \mid i,j \in V, i \neq j\} - E$; that is, $\overline{G}$ has the same set of vertices as G, but an edge e exists is $\overline{G}$ if and only if it does not exist in G.

(a) Suppose G has v vertices and e edges. How many edges does $\overline{G}$ have?

(b) Prove that for any graph with at least 13 vertices, G being planar implies that $\overline{G}$ is non-planar.

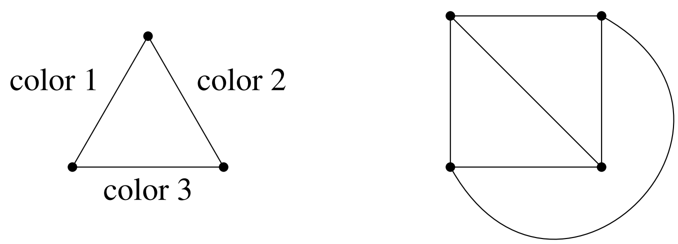
(c) Now consider the converse of the previous part, i.e., for any graph G with at least 13 vertices, if $\overline{G}$ is non-planar, then G is planar. Construct a counterexample to show that the converse does not hold.

[*Hint*: Recall that if a graph contains a copy of K_5, then it is non-planar. Can this fact be used to construct a counterexample?]

5 Edge Colorings

Note 5

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.



(a) Show that the 4 vertex complete graph, K_4 (shown above on the right), can be edge colored with 3 colors. (You may use the numbers $1,2,3$ for colors.)

(b) Let $d \geq 1$. Prove that any graph with maximum degree d can be edge colored with $2d-1$ colors.

(c) Prove that a tree can be edge colored with d colors where d is the maximum degree of any vertex.

[*Hint*: Different problems sometimes are easier to induct over different parameters even when the problems are very similar.]

6 Touring Hypercube

Note 5

In the lecture, you have seen that if G is a hypercube of dimension n, then

- The vertices of G are the binary strings of length n.
- u and v are connected by an edge if they differ in exactly one bit location.

A *Hamiltonian cycle* of a graph (with ≥ 2 vertices) is a cycle that visits every vertex exactly once (except that the start and end vertices are the same).

(a) Prove that a hypercube has an Eulerian tour if and only if n is even.

(b) Prove that every hypercube of dimension $n \geq 2$ has a Hamiltonian cycle.

7 Modular Practice

Note 6

(a) Compute $13^{2026} \pmod{12}$.

(b) Compute $7^{62} \pmod{11}$.

(c) Find $\gcd(2n+1, 3n+2)$, where n is a positive integer.

(d) Solve for y: $10y + 5 \equiv 7 \pmod{13}$.

(e) Solve the system of equations for a and b: $5a + 4b \equiv 0 \pmod{7}$ and $2a + b \equiv 4 \pmod{7}$.

(f) Prove that $3x + 12 \equiv 4 \pmod{21}$ does not have a solution.

8 Wilson's Theorem

Note 6

Wilson's Theorem states the following is true if and only if p is prime:

$$(p-1)! \equiv -1 \pmod{p}.$$

Prove both directions (it holds if AND only if p is prime).

[*Hint for the if direction*: Consider rearranging the terms in $(p-1)! = 1 \cdot 2 \cdot \cdots \cdot (p-1)$ to pair up terms with their inverses, when possible. What terms are left unpaired?]

[*Hint for the only if direction*: If p is composite, then it has some prime factor q. What can we say about $(p-1)! \pmod{q}$?]