

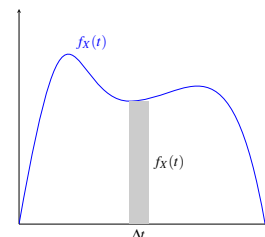
Continuous Probability Intro I

In discrete probability, we are only concerned with RVs that take on countably many values; now, in continuous probability, we are interested in RVs that take on *uncountably* many values. Here, the most important difference is that $\mathbb{P}[X = k] = 0$ for any k , but we can have $\mathbb{P}[X \in (a, b)] > 0$.

This motivates the **probability density function (PDF)**, denoted as $f_X(t)$.

Recall from physics that in 1D, density = mass/length; we define *probability density* similarly, as probability/length. So the area of the rectangle, $f_X(t)\Delta t$ (a product of density and length), is a probability. Generalizing this idea to find the area under the curve, we can now find probabilities from the PDF:

$$\mathbb{P}[a \leq X \leq b] = \int_a^b f_X(t) dt.$$



From here, we define the **cumulative distribution function (CDF)** as

$$F_X(t) = \mathbb{P}[X \leq t] = \int_{-\infty}^t f_X(u) du.$$

From the fundamental theorem of calculus, we have $\frac{d}{dt}F_X(t) = f_X(t)$; the derivative of the CDF is the PDF.

Properties of PDF and CDF:

- $f_X(t) \geq 0$, and $\int_{-\infty}^{\infty} f_X(t) dt = 1$
- $F_X(t)$ must be non-decreasing, with $\lim_{t \rightarrow \infty} F_X(t) = 1$ and $\lim_{t \rightarrow -\infty} F_X(t) = 0$

The **joint density** $f(x, y)$ satisfies $\mathbb{P}[a \leq X \leq b, c \leq Y \leq d] = \int_a^b \int_c^d f(x, y) dy dx$ and integrates to 1.

Other probability concepts follow naturally from these definitions; the only major difference from discrete probability is that sums turn into integrals, and $\mathbb{P}[X = t]$'s turn into $f_X(t)$'s. For example,

	Discrete	Continuous
Expectation	$\mathbb{E}[X] = \sum_t t \cdot \mathbb{P}[X = t]$	$\mathbb{E}[X] = \int_{-\infty}^{\infty} t \cdot f_X(t) dt$
LOTUS	$\mathbb{E}[g(X)] = \sum_t g(t) \cdot \mathbb{P}[X = t]$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(t) \cdot f_X(t) dt$
Total Probability	$\mathbb{P}[A] = \sum_{t=1}^n \mathbb{P}[A X = t] \mathbb{P}[X = t]$	$\mathbb{P}[A] = \int_{-\infty}^{\infty} \mathbb{P}[A X = t] f_X(t) dt$
Marginal Distribution/Density	$\mathbb{P}[X = x] = \sum_y \mathbb{P}[X = x, Y = y]$	$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy$

Exponential Distribution: $X \sim \text{Exponential}(\lambda)$, the continuous analog to the geometric distribution; it models the amount of time needed to wait until a success, where the rate of success is λ .

$$f_X(t) = \lambda e^{-\lambda t} \quad F_X(t) = 1 - e^{-\lambda t} \quad \mathbb{E}[X] = \frac{1}{\lambda} \quad \text{Var}(X) = \frac{1}{\lambda^2}$$

Similar to the geometric distribution, the exponential distribution also has the memoryless property:

$$\mathbb{P}[X > m + n | X > m] = \mathbb{P}[X > n].$$

1 Continuous Intro

Note 20

(a) Is $g(x)$ a valid probability density function? Why or why not? Is $g(x)$ a valid CDF? Why or why not?

$$g(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) Let X be a continuous random variable with CDF $F_X(x)$. Calculate the PDF $f_X(x)$, $\mathbb{E}[X]$ and $\text{Var}(X)$.

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{\ell}, & 0 \leq x \leq \ell, \\ 1, & x > \ell \end{cases}$$

2 Darts Again

Note 20

Edward and Khalil are playing darts on a circular dartboard.

Edward's throws are uniformly distributed over the entire dartboard, which has a radius of 10 inches. Khalil has good aim (but his throws may land outside of the dartboard); the distance of his throws from the center of the dartboard follows an exponential distribution with parameter $\frac{1}{2}$.

Say that Edward and Khalil both throw one dart at the dartboard. Let X be the distance of Edward's dart from the center, and Y be the distance of Khalil's dart from the center of the dartboard. What is $\mathbb{P}[X < Y]$, the probability that Edward's throw is closer to the center of the board than Khalil's? Leave your answer in terms of an unevaluated integral.

[Hint: X is not uniform over $[0, 10]$. Solve for the distribution of X by first computing the CDF of X , $\mathbb{P}[X \leq x]$.]

3 Continuous Joint Densities

Note 20

The joint probability density function of two random variables X and Y is given by $f(x,y) = Cxy$ for $0 \leq x \leq 1, 0 \leq y \leq 2$, and 0 otherwise, where C is a constant.

(a) Find the value of C that ensures that $f(x,y)$ is indeed a probability density function.

(b) Find $f_X(x)$, the marginal density function of X .

(c) Calculate $\mathbb{E}[XY]$.

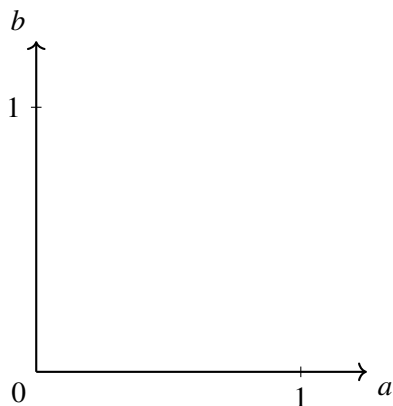
4 Lunch Meeting

Note 20

Alice and Bob agree to try to meet for lunch between 12 PM and 1 PM at their favorite sushi restaurant. Being extremely busy, they are unable to specify their arrival times exactly, and can say only that each of them will arrive (independently) at a time that is uniformly distributed within the hour. In order to avoid wasting precious time, if the other person is not there when they arrive they agree to wait exactly fifteen minutes before leaving.

Let A be the time that Alice arrives and B be the time that Bob arrives, both measured in hours after 12 PM (so A and B are each uniformly distributed on $[0, 1]$).

- (a) Provide a sketch of the joint distribution of (A, B) on the axes below, where the horizontal axis a is the value of A and the vertical axis b is the value of B . For which region of the graph will Alice and Bob actually meet?



- (b) Based on your sketch, what is the probability that they will actually meet for lunch?