Concentration Inequalities Intro

Markov's Inequality: For any nonnegative random variable X and $t > 0$,

$$\mathbb{P}[X \geq t] \leq \frac{\mathbb{E}[X]}{t}.$$

Chebyshev's Inequality: For any random variable X and $c > 0$,

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq c] \leq \frac{\operatorname{Var}(X)}{c^2}.$$

Law of Large Numbers: Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with mean μ and variance σ^2. We have the following:

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \mu$$

$$\operatorname{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{\sigma^2}{n}.$$

Applying Chebyshev's inequality on the sample mean $\frac{1}{n}\sum_{i=1}^{n} X_i$, we have that

$$\mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \varepsilon\right] \leq \frac{\sigma^2}{n\varepsilon^2}$$

which means that as $n \to \infty$, the probability of the sample mean deviating from the true mean by any $\varepsilon > 0$ approaches zero.

1 Probabilistic Bounds

Note 19

For each part below, use Markov's or Chebyshev's inequality to bound the given probability. Remember that Markov's inequality requires a *non-negative* random variable, and Chebyshev's inequality bounds the absolute deviation from the mean, $|X - \mathbb{E}[X]|$.

Hint: to use Markov's inequality, first define a new non-negative random variable from the one given.

(a) Let W be a random variable with $W \geq -5$ and $\mathbb{E}[W] = -3$. Find an upper bound for $\mathbb{P}[W \geq -1]$.

(b) Let X be a random variable with $\mathbb{E}[X] = 2$ and $\mathrm{Var}(X) = 9$, whose value never exceeds 10. Show that $\mathbb{P}[X \leq 1] \leq 8/9$.

(c) For the same random variable X from the previous part, show that $\mathbb{P}[X \geq 6] \leq 9/16$.

(d) You roll a fair die 100 times. Let Z be the sum of the numbers that appear. Compute $\mathrm{Var}(Z)$, then use Chebyshev's inequality to bound the probability that Z is greater than 400 or less than 300.

2 Vegas

Note 19

On the planet Vegas, everyone carries a coin. Many people are honest and carry a fair coin (heads on one side and tails on the other), but a fraction p of them cheat and carry a trick coin with heads on both sides. You want to estimate p with the following experiment: you pick a random sample of n people and ask each one to flip their coin. Assume that each person is independently likely to carry a fair or a trick coin.

(a) Let X be the proportion of coin flips which are heads. Find $\mathbb{E}[X]$.

(b) Given the results of your experiment, how should you estimate p? (*Hint:* we're looking for an estimate $\hat{p}$ such that $\mathbb{E}[\hat{p}] = p$. Think about how you could find p by rewriting our expression for $\mathbb{E}[X]$.)

(c) Find the smallest n, the number of people you ask, for which Chebyshev's inequality *guarantees* that your estimate $\hat{p}$ is within 0.05 of the true value of p with probability greater than 0.95 — that is, guarantees that $\mathbb{P}[|\hat{p} - p| \leq 0.05] > 0.95$. (Note that this is not necessarily the smallest n for which the guarantee actually holds; Chebyshev's inequality is only an upper bound on the failure probability.)

3 Working with the Law of Large Numbers

Note 19

(a) A fair coin is tossed multiple times and you win a prize if there are more than 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

(b) A fair coin is tossed multiple times and you win a prize if there are more than 40% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

(c) A fair coin is tossed multiple times and you win a prize if there are between 40% and 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

(d) A fair coin is tossed multiple times and you win a prize if there are exactly 50% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.