Combinations of Events Intro

Note 14

Independence: Two events are independent if the following (equivalent) conditions are satisfied. The second definition is probably more intuitive - B happening does not affect the probability of A happening.

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$$
$$\mathbb{P}[A \mid B] = \mathbb{P}[A]$$

Product rule: We can find the probability of an intersection of events by enforcing an "ordering" of these events. Here, each successive conditional probability in the product finds the probability of the next event, *conditioned* on all prior events occurring:

$$\mathbb{P}[A_1 \cap A_2 \cap \cdots \cap A_n] = \mathbb{P}[A_1]\,\mathbb{P}[A_2 \mid A_1]\,\mathbb{P}[A_3 \mid A_1 \cap A_2] \cdots \mathbb{P}[A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}].$$

Note that this is a generalization of the definition of conditional probability:

$$\mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_1]\,\mathbb{P}[A_2 \mid A_1].$$

Union Bound: The probability that at least one of the events $A_1, A_2, \ldots, A_n$ occurs is at most the sum of the probabilities of the individual events:

$$\mathbb{P}[A_1 \cup A_2 \cup \cdots \cup A_n] \le \mathbb{P}[A_1] + \mathbb{P}[A_2] + \cdots + \mathbb{P}[A_n]$$
$$\mathbb{P}[\bigcup_{i=1}^{n} A_i] \le \sum_{i=1}^{n} \mathbb{P}[A_i]$$

with equality when the A_i's are disjoint.

Inclusion-Exclusion: We also have PIE in probability! As with counting, the minimal example is:

$$\mathbb{P}[A_1 \cup A_2] = \mathbb{P}[A_1] + \mathbb{P}[A_2] - \mathbb{P}[A_1 \cap A_2].$$

And the general case is similar, albeit a bit notation-heavy:

$$\mathbb{P}[\bigcup_{i=1}^{n} A_i] = \sum_{i} \mathbb{P}[A_i] - \sum_{i<j} \mathbb{P}[A_i \cap A_j] + \sum_{i<j<k} \mathbb{P}[A_i \cap A_j \cap A_k] - \cdots + (-1)^{n-1}\,\mathbb{P}[A_1 \cap A_2 \cdots \cap A_n]$$

Notice that the Union Bound is just PIE without any correction terms for intersections of events. In practice, the Union Bound is one of the most beloved inequalities in probability because it is a quick, simple, and unconditional upper bound of the exact PIE expression.

1 Probability Potpourri

Note 13
Note 14

Provide brief justification for each part.

(a) For two events A and B in the same probability space, show that $\mathbb{P}[A \setminus B] \geq \mathbb{P}[A] - \mathbb{P}[B]$.

(b) Suppose $\mathbb{P}[D \mid C] = \mathbb{P}[D \mid \overline{C}]$, where $\overline{C}$ is the complement of C. Prove that D is independent of C.

(c) If A and B are disjoint, does that imply they're independent?

2 Mario's Coins

Note 14

Mario owns three identical-looking coins. One coin shows heads with probability $1/4$, another shows heads with probability $1/2$, and the last shows heads with probability $3/4$.

(a) Mario randomly picks a coin and flips it. He then picks one of the other two coins and flips it. Let X_1 and X_2 be the events of the 1st and 2nd flips showing heads, respectively. Are X_1 and X_2 independent? Please prove your answer.

(b) Mario randomly picks a single coin and flips it twice. Let Y_1 and Y_2 be the events of the 1st and 2nd flips showing heads, respectively. Are Y_1 and Y_2 independent? Please prove your answer.

(c) Mario arranges his three coins in a row. He flips the coin on the left, which shows heads. He then flips the coin in the middle, which shows heads. Finally, he flips the coin on the right. What is the probability that it also shows heads?

3 Aces

Note 14

Consider a standard 52-card deck of cards, which has 4 suits (hearts, diamonds, clubs, and spades) with 13 cards in each suit. Each suit has one ace. Hearts and diamonds are red, while clubs and spades are black.

(a) Find the probability of getting an ace or a red card, when drawing a single card.

(b) Find the probability of getting an ace or a spade, but not both, when drawing a single card.

(c) Find the probability of getting the ace of diamonds when drawing a 5 card hand.

(d) Find the probability of getting exactly 2 aces when drawing a 5 card hand.

(e) Find the probability of getting at least 1 ace when drawing a 5 card hand.

(f) Find the probability of getting at least 1 ace or at least 1 heart when drawing a 5 card hand.