CS 70 Discrete Mathematics and Probability Theory
Summer 2026 Tate **DIS 4B**

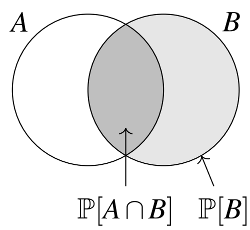
Conditional Probability Intro

Note 14

Conditional Probability: Probability of event A, *given* that event B has happened (implying that $P[B] > 0$);

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}.$$

Think of like restricting our sample space:



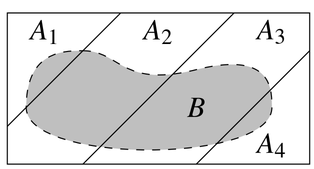
Bayes' Rule[1]: A consequence of conditional probability - notice $\mathbb{P}[A \cap B] = \mathbb{P}[A \mid B]\,\mathbb{P}[B] = \mathbb{P}[B \mid A]\,\mathbb{P}[A]$, so

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[B \cap A]}{\mathbb{P}[B]} = \frac{\mathbb{P}[B \mid A]\,\mathbb{P}[A]}{\mathbb{P}[B]}.$$

Total Probability Rule: If disjoint events $A_1, \ldots, A_n$ form a partition on the sample space Ω (meaning that each outcome in Ω belongs to exactly one of $A_1, \ldots, A_n$.), we then have

$$\mathbb{P}[B] = \sum_{i=1}^{n} \mathbb{P}[B \cap A_i] = \sum_{i=1}^{n} \mathbb{P}[B \mid A_i]\,\mathbb{P}[A_i].$$

Visually, we're splitting an event into partitions and looking at each intersection individually:



Most notably, we can always condition on the partition $\{A, \overline{A}\}$ to obtain the expression:

$$\mathbb{P}[B] = \mathbb{P}[B \cap A] + \mathbb{P}[B \cap \overline{A}] = \mathbb{P}[B|A]\,\mathbb{P}[A] + \mathbb{P}[B|\overline{A}]\,\mathbb{P}[\overline{A}].$$

This formula is useful when we don't know the *total probability* $\mathbb{P}[B]$ (I don't care whether A happens or not) but know the conditioned probabilities $\mathbb{P}[B|A]$ and $\mathbb{P}[B|\overline{A}]$.

[1] [Optional Reading] Another way to think about Bayes' rule is *updating your belief.* You start with some probability, $\mathbb{P}[A]$. Then, you observe some other event B. Now, you are considering $\mathbb{P}[A|B]$, which is the probability of A happening *given that you know that B happened*. This may alter your belief about the event A happening (either more likely or less likely). And according to Bayes' rule, this act of "updating based on new information" amounts to multiplying $\mathbb{P}[A]$ by some number $\frac{\mathbb{P}[B|A]}{\mathbb{P}[B]}$ (Notice that if A and B are independent, $\frac{\mathbb{P}[B|A]}{\mathbb{P}[B]} = 1$, so your belief stays the same. Nice!). This way of thinking is often called **Bayesian** or **Bayesian Inference**, and it is the pillar of modern machine learning and inference algorithms (given that I know these data, how can I best predict the future?).

1 Poisoned Smarties

Note 14

Supposed there are 3 people who are all owners of their own Smarties factories. Burr Kelly, being the brightest and most innovative of the owners, produces considerably more Smarties than her competitors and has a commanding 50% of the market share. Yousef See, who inherited her riches, lags behind Burr and produces 40% of the world's Smarties. Finally Stan Furd, brings up the rear with a measly 10%. However, a recent string of Smarties related food poisoning has forced the FDA to investigate these factories to find the root of the problem. Through her investigations, the inspector found that 2 Smarties out of every 100 at Kelly's factory was poisonous. At See's factory, 5% of Smarties produced were poisonous. And at Furd's factory, the probability a Smarty was poisonous was 0.1.

(a) What is the probability that a randomly selected Smarty will be safe to eat?

(b) If we know that a certain Smarty didn't come from Burr Kelly's factory, what is the probability that this Smarty is poisonous?

(c) If a randomly selected Smarty is poisonous, what is the probability it came from Stan Furd's Smarties Factory?

2 Symmetry

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In this problem, we will walk you through the idea of *symmetry* and its formal justification. Consider an experiment where you have a bag with m red marbles and $n-m$ blue marbles. You draw marbles from the bag, one at a time without replacement until the bag is empty.

(a) Define the sample space Ω. (No need to write out every element, a brief description is fine). Is this a uniform probability space?

(b) What is the probability that the first marble you draw is red?

(c) Suppose you've drawn all but the final marble, setting each marble aside as you draw it *without looking at it*. We want to find the probability that the final marble left in the bag will be red.

Let A be the event containing outcomes where the first marble is red, and let B be the event containing outcomes where the final marble is red. Describe, in English, a bijective function $f: A \to B$ mapping outcomes in A to outcomes in B, and explain why it is a bijection. Note that there can be multiple valid bijections. A bijection is a one-to-one mapping.

(d) Use the previous parts to find the probability that the final marble will be red.

(e) You repeat the experiment. Find the probability that the last two marbles you draw will be red.

(f) You repeat the experiment again, but this time you see that the first marble you draw is red. Find the probability that the second-to-last marble you draw will also be red.

3 Pairwise Independence

Note 14

Recall that the events A_1, A_2, and A_3 are *pairwise independent* if for all $i \neq j$, A_i is independent of A_j. However, pairwise independence is a weaker statement than *mutual independence*, which requires the additional condition that $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \mathbb{P}[A_1]\,\mathbb{P}[A_2]\,\mathbb{P}[A_3]$.

Suppose you roll two fair six-sided dice. Let A_1 be the event that the first die lands on 1, let A_2 be the event that the second die lands on 6, and let A_3 be the event that the two dice sum to 7.

(a) Compute $\mathbb{P}[A_1]$, $\mathbb{P}[A_2]$, and $\mathbb{P}[A_3]$.

(b) Are A_1 and A_2 independent?

(c) Are A_2 and A_3 independent?

(d) Are A_1, A_2, and A_3 pairwise independent?

(e) Are A_1, A_2, and A_3 mutually independent?