1 Countability Intro

Note 11

A function $f : A \to B$ maps elements from set A to set B.

f is *injective* if it maps distinct elements to distinct elements, and *surjective* if every element in B is mapped to by some element in A. If f is both injective and surjective, it is *bijective*, and the sets A and B are said to have the same *cardinality* (size). The cardinality of a set is denoted by $|A|$.

f is bijective if and only if there exists an inverse function $f^{-1} : B \to A$ such that $f^{-1}(f(a)) = a$ for all $a \in A$ and $f(f^{-1}(b)) = b$ for all $b \in B$.

Countability: Formal notion of different kinds of infinities.

- *Countable*: able to enumerate in a list (possibly finite, possibly infinite)
- *Countably infinite*: able to enumerate in an infinite list; that is, there is a bijection with $\mathbb{N}$.

  To show that there is a bijection, the *Cantor–Bernstein theorem* says that it is sufficient to find two injections, $f : S \to \mathbb{N}$ and $g : \mathbb{N} \to S$. Intuitively, this is because an injection $f : S \to \mathbb{N}$ means $|S| \leq |\mathbb{N}|$, and an injection $g : \mathbb{N} \to S$ means $|\mathbb{N}| \leq |S|$; together, we have $|\mathbb{N}| = |S|$.
- *Uncountably infinite*: a set with a cardinality larger than $|\mathbb{N}|$. Therefore, there is no bijection with $\mathbb{N}$ and the set can't be listed out.

  Use *Cantor diagonalization* to prove uncountability through contradiction; the classic example is the set of reals in $[0,1]$:

$$\begin{array}{c|l} \mathbb{N} & [0,1] \\ \hline 0 & 0\,.\,7\;3\;2\;0\;5\;0\;\cdots \\ 1 & 0\,.\,4\;1\;4\;2\;1\;3\;\cdots \\ 2 & 0\,.\,6\;1\;8\;0\;3\;3\;\cdots \\ 3 & 0\,.\,1\;8\;2\;8\;1\;8\;\cdots \\ 4 & 0\,.\,1\;4\;1\;5\;9\;2\;\cdots \\ 5 & 0\,.\,5\;7\;7\;2\;1\;5\;\cdots \\ \vdots & \qquad\vdots \\ \hline ? & 0\,.\,8\;2\;9\;9\;1\;6\;\cdots \end{array}$$

  To have a bijection with $\mathbb{N}$, we must be able to assign a natural number to every real number, as above. In other words, every real number must eventually appear on this (infinite) list. However, if we change the digits along the diagonal and compile that into a new real number, it is different from every single element in the list in at least one place, so it's not in the list—this is a **contradiction**, and it implies that there is at least one real number "left over". Hence, the cardinality of $\mathbb{R}$ is strictly greater than $\mathbb{N}$!

Sometimes it can be easier to prove countability/uncountability through bijections with other countable/uncountable sets respectively. Common countable sets include $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{N} \times \mathbb{N}$, finite length bitstrings, etc. Common uncountable sets include $[0,1]$, $\mathbb{R}$, infinite length bitstrings, etc.

(a) Read through the previous page's description of Cantor diagonalization. In your own words, why do we know that the number created by changing the digits along the diagonal is not in the list?

(b) True or false: the new number created is a real number in $[0,1]$ that is not on the list.

(c) Your friend is confused about how Cantor diagonalization doesn't apply to the set of natural numbers. They argue that natural numbers can be thought of as an infinite length string of digits, by padding each number with an infinite number of zeroes to the left. If we then assume by contradiction that we can list out the set of natural numbers with the padded 0's, we can change the digits along a diagonal, to create a new natural number not in the list.

$$\begin{array}{c|cccccc}
0 & \cdots & 0 & 0 & 0 & 0 & 1 \\
1 & \cdots & 0 & 1 & 2 & 3 & 4 \\
2 & \cdots & 5 & 2 & 8 & 2 & 3 \\
3 & \cdots & 9 & 4 & 3 & 2 & 1 \\
\vdots & & & & \vdots & & \\
\hline
? & \cdots & 1 & 5 & 9 & 4 & 2
\end{array}$$

What is wrong with this argument?

2 Countability: True or False

Note 11

(a) The set of all irrational numbers $\mathbb{R} \backslash \mathbb{Q}$ (i.e. real numbers that are not rational) is uncountable.

(b) The set of real solutions for the equation $x + y = 1$ is countable.

For any two functions $f : Y \to Z$ and $g : X \to Y$, let their composition $f \circ g : X \to Z$ be given by $f \circ g = f(g(x))$ for all $x \in X$. Determine if the following statements are true or false.

(c) f and g are injective (one-to-one) $\implies$ $f \circ g$ is injective (one-to-one).

(d) f is surjective (onto) $\implies$ $f \circ g$ is surjective (onto).

3 Counting Cartesian Products

Note 11

For two sets A and B, define the cartesian product as $A \times B = \{(a,b) : a \in A, b \in B\}$.

(a) Given two countable sets A and B, prove that $A \times B$ is countable.

(b) Given a finite number of countable sets $A_1, A_2, \ldots, A_n$, prove that

$$A_1 \times A_2 \times \cdots \times A_n$$

is countable.

(c) Consider a countably infinite number of finite sets: $B_1, B_2, \ldots$ for which each set has at least 2 elements. Prove that $B_1 \times B_2 \times \cdots$ is uncountable.

4 Counting Functions

Note 11

Are the following sets countable or uncountable? Prove your claims.

(a) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-decreasing. That is, $f(x) \leq f(y)$ whenever $x \leq y$.

(b) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-increasing. That is, $f(x) \geq f(y)$ whenever $x \leq y$.