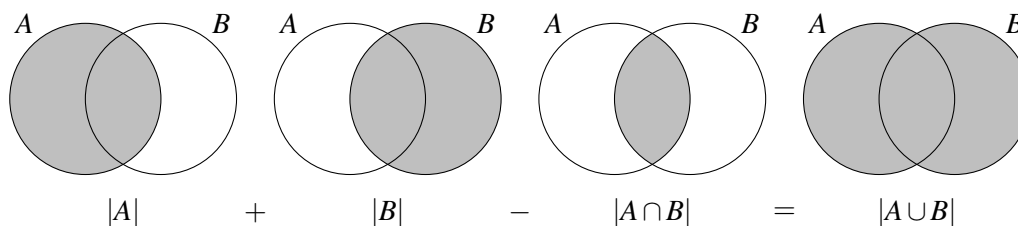


Counting Intro II

Note 10

Inclusion-exclusion: With two sets,



With more sets,

$$\begin{aligned}
 |A_1 \cup A_2 \cup \dots \cup A_n| &= |A_1| + |A_2| + \dots + |A_n| \\
 &\quad - |A_1 \cap A_2| - |A_1 \cap A_3| - \dots - |A_i \cap A_j| - \dots - |A_{n-1} \cap A_n| \\
 &\quad + |A_1 \cap A_2 \cap A_3| + \dots + |A_i \cap A_j \cap A_k| + \dots + |A_{n-2} \cap A_{n-1} \cap A_n| \\
 &\quad \dots \\
 \left| \bigcup_{i=1}^n A_i \right| &= \sum_{k=1}^n (-1)^{k-1} \sum_{\substack{S \subseteq \{1, \dots, n\} \\ |S|=k}} \left| \bigcap_{i \in S} A_i \right|
 \end{aligned}$$

That is, for each size k , iterate through all ways of picking k sets from $\{A_1, \dots, A_n\}$, and alternate between adding and subtracting the sizes of their intersection.

Combinatorial proofs: A technique for proving combinatorial identities. There should be very little math involved (usually none): use two different ways of counting the same scenario. One way should correspond to the left-hand side of the equality, and the other way should correspond to the right-hand side of the equality. The fact that we're counting the same scenario means that the two sides are equal.

1 Inclusion and Exclusion

Note 10

What is the total number of positive integers strictly less than 60 that are also coprime to 60?

2 CS70: The Musical

Note 10

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

- (a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

- (b) Edward would now like to select a crew out of n people. Use this to provide a combinatorial argument that proves the following identity: (this is called Pascal's Identity)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

- (c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity:

$$\sum_{k=1}^n k \binom{n}{k} = n2^{n-1}$$

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity:

$$\sum_{k=j}^n \binom{n}{k} \binom{k}{j} = 2^{n-j} \binom{n}{j}.$$

3 Fibonacci Fashion

Note 10 You have n accessories in your wardrobe, and you'd like to plan which ones to wear each day for the next t days. As a student of the Elegant Etiquette Charm School, you know it isn't fashionable to wear the same accessories multiple days in a row. (Note that the same goes for clothing items in general). Therefore, you'd like to plan which accessories to wear each day represented by subsets $S_1, S_2, \dots, S_t$, where $S_1 \subseteq \{1, 2, \dots, n\}$ and for $2 \leq i \leq t$, $S_i \subseteq \{1, 2, \dots, n\}$ and S_i is disjoint from S_{i-1} .

(a) For $t \geq 1$, prove that there are F_{t+2} binary strings of length t with no consecutive zeros (assume the Fibonacci sequence starts with $F_0 = 0$ and $F_1 = 1$).

(b) Use a combinatorial proof to prove the following identity, which, for $t \geq 1$ and $n \geq 0$, gives the number of ways you can create subsets of your n accessories for the next t days such that no accessory is worn two days in a row:

$$\sum_{x_1 \geq 0} \sum_{x_2 \geq 0} \cdots \sum_{x_t \geq 0} \binom{n}{x_1} \binom{n-x_1}{x_2} \binom{n-x_2}{x_3} \cdots \binom{n-x_{t-1}}{x_t} = (F_{t+2})^n.$$

(You may assume that $\binom{a}{b} = 0$ whenever $a < b$.)